# Direct generation of spinning Einstein–Maxwell fields from static fields*

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**Abstract.** I present a new method to generate rotating solutions of the Einstein–Maxwell equations from static solutions, and briefly discuss its general properties.

The four–dimensional stationary Einstein–Maxwell equations are well–known to be invariant under an SU(2,1) group of transformations [1]. Applied to asymptotically flat solutions, these transformations map monopole solutions into monopole solutions (e.g. the Schwarzschild solution into the Reissner–Nordström solution), so that they cannot be used to transform a static monopole solution into a rotating monopole–dipole solution, such as the Kerr solution. The situation becomes different in the case of stationary axisymmetric solutions, with two commuting Killing vectors. By combining the invariance transformations associated with a given direction in the Killing 2–plane with rotations in this plane, the infinite–dimensional Geroch group emerges [2]. These transformations allow in principle the generation of all solutions of the stationary axisymmetric Einstein–Maxwell problem, which is thus completely integrable. This generation of stationary axisymmetric solutions can be achieved in a variety of manners, for instance by exponentiation of the infinitesimal group action [3] or by the inverse–scattering transform method [4].

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In this talk, I shall present a new, simple method to generate asymptotically flat spinning Einstein–Maxwell fields from asymptotically flat static fields by using finite combinations of SU(2,1) transformations and global coordinate transformations mixing the two Killing vectors [5]. After recalling briefly the Ernst approach to the reduction of the stationary Einstein–Maxwell problem, I shall describe this direct rotation–generating transformation, give some examples of its application, and discuss its general properties.

Stationary Einstein–Maxwell fields without matter sources may be parametrized by the metric and electromagnetic fields

$$\begin{aligned} ds^2 &= f\,(dt - \omega_i dx^i)^2 - f^{-1}\,h_{ij}\,dx^i dx^j\,, \\ F_{i0} &= \partial_i v\,, \qquad F^{ij} = f\,h^{-1/2}\epsilon^{ijk}\partial_k u\,, \end{aligned} \tag{1}$$

where the various fields depend only on the space coordinates $x^i$. In the case of axisymmetric fields, it is often convenient to choose Weyl coordinates $\rho$, $z$, $\varphi$ such that

$$\omega_i\,dx^i \equiv \omega(\rho,z)\,d\varphi\,, \qquad h_{ij}\,dx\,dx^j = \mathrm{e}^{2k(\rho,z)}(d\rho^2 + dz^2) + \rho^2\,d\varphi^2\,. \tag{2}$$

The stationary Einstein–Maxwell equations may be reduced to the three–dimensional Ernst equations [6]

$$\begin{aligned} f\nabla^2\mathcal{E} &= \nabla\mathcal{E}\cdot(\nabla\mathcal{E} + 2\overline{\psi}\nabla\psi)\,, \\ f\nabla^2\psi &= \nabla\psi\cdot(\nabla\mathcal{E} + 2\overline{\psi}\nabla\psi)\,, \\ f^2 R_{ij}(h) &= \mathrm{Re}\left[\frac{1}{2}\mathcal{E}_{,(i}\,\overline{\mathcal{E}}_{,j)} + 2\psi\mathcal{E}_{,(i}\,\overline{\psi}_{,j)} - 2\mathcal{E}\psi_{,(i}\,\overline{\psi}_{,j)}\right]\,, \end{aligned} \tag{3}$$

where the scalar products and covariant Laplacian are computed with the reduced spatial metric $h_{ij}$, the complex Ernst potentials are defined by

$$\mathcal{E} = f + i\chi - \overline{\psi}\psi\,, \qquad \psi = v + iu\,. \tag{4}$$

and $\chi$ is the twist potential

$$\partial_i\chi = -f^2\,h^{-1/2}h_{ij}\,\epsilon^{jkl}\,\partial_k\omega_l + 2(u\partial_i v - v\partial_i u)\,. \tag{5}$$

These equations are invariant under an SU(2,1) group of transformations [1].

The direct rotation–generating transformation is the product

$$\Sigma = \Pi^{-1}\,\mathcal{R}\,\Pi \tag{6}$$

of three successive transformations, two "vertical" transformations $\Pi\,,\Pi^{-1}\in \mathrm{SU}(2,1)$ acting on the potential space, and a "horizontal" global coordinate transformation $\mathcal{R}$ acting on the Killing 2–plane. The transformation $\Pi$ is the SU(2,1) involution $(\mathcal{E},\psi,h_{ij})\leftrightarrow(\hat{\mathcal{E}},\hat{\psi},\hat{h}_{ij})$ with

$$\hat{\mathcal{E}} = \frac{-1+\mathcal{E}+2\psi}{1-\mathcal{E}+2\psi}\,, \quad \hat{\psi} = \frac{1+\mathcal{E}}{1-\mathcal{E}+2\psi}\,, \quad \hat{h}_{ij} = h_{ij}\,. \tag{7}$$

Consider the Schwarzschild solution, written in prolate spheroidal coordinates [7]

$$ds^2 = f dt^2 - f^{-1}m^2\,[dx^2 + (x^2-1)(d\theta^2 + \sin^2\theta\, d\varphi^2)]\,, \qquad f = (x-1)/(x+1) \tag{8}$$

(the coordinate $x$ is related to the "standard" radial coordinate $r$ by $x = (r-m)/m$), with the Ernst potentials $\mathcal{E} = (x-1)/(x+1)$, $\psi = 0$. The action of $\Pi$ leads, after a trivial rescaling of the time coordinate, to the non asymptotically flat Bertotti–Robinson (BR) solution [8]

$$d\hat{s}^2 = m^2\left[x^2-1)\,d\tau^2 - \frac{dx^2}{x^2-1} - \frac{dy^2}{1-y^2} - (1-y^2)\,d\varphi^2\right]\,, \tag{9}$$

with the transformed Ernst potentials $\hat{\mathcal{E}} = -1$, $\hat{\psi} = x$. More generally, if the initial Ernst potentials are asymptotically monopole, the transformation $\Pi$ leads to asymptotically BR–like potentials.

The global coordinate transformation $\mathcal{R}(\Omega,\gamma)$ is the product of the transformation to a uniformly rotating frame and of a time dilation,

$$d\varphi = d\varphi' + \Omega\gamma\, dt'\,, \qquad dt = \gamma\, dt'\,. \tag{10}$$

In the case of electrostatic solutions with $\hat{\mathcal{E}}$ and $\hat{\psi}$ real, ($\hat{\omega} = 0$), the frame rotation gives rise to an induced gravimagnetic field $\hat{\omega}'$ as well as to an induced magnetic field. However this transformation does not modify the leading asymptotic behavior of the BR metric or of asymptotically BR–like metrics. Because of this last property, the last transformation $\Pi^{-1}$ in (6) then leads to asymptotically flat, but complex, Ernst potentials corresponding to a monopole–dipole solution.

As shown in [5] the transformation $\Sigma$ leads, for the special choice $\gamma = (1 + m^2\Omega^2)^{-1/2}$, from the Schwarzschild solution to the Kerr solution. For other values of $\gamma$ the Kerr–Newman family of solutions is obtained. Another example is that of the Voorhees–Zipoy family of static vacuum solutions [7], depending on a real parameter $\delta$. The action of the transformation $\Sigma$ leads to new rotating solutions [5] with dipole magnetic moment and quadrupole electric moment which are different from the discrete ($\delta$ integer) Tomimatsu–Sato [9] family of vacuum rotating

solutions. Yet another example is the generation of spinning ring solutions of the Einstein–Maxwell equations from static ring wormhole solutions [10].

In the case of a generic axisymmetric electrostatic solution ($\mathcal{E}$, $\psi$ real), the transformed BR–like Ernst potentials $\hat{\mathcal{E}}$, $\hat{\psi}$ satisfy the real Ernst equations,

$$\nabla(\rho \hat{f}^{-1}\nabla\hat{\mathcal{E}}) = 0\,, \qquad \nabla(\rho \hat{f}^{-1}\nabla\hat{\psi}) = 0\,, \tag{11}$$

which imply the existence of dual Ernst potentials $\hat{\mathcal{F}}$, $\hat{\phi}$ such that

$$\hat{\mathcal{F}}_{,m} = \rho \hat{f}^{-1}\,\epsilon_{mn}\hat{\mathcal{E}}_{,n}\,, \quad \hat{\phi}_{,m} = \rho \hat{f}^{-1}\,\epsilon_{mn}\hat{\psi}_{,n} \tag{12}$$

($m$,$n$ = 1,2, with $x^1 = \rho$, $x^2 = z$). It may then be shown that the transformation $\mathcal{R}$ with $\gamma = 1$ transforms the potentials ($\hat{\mathcal{E}}$, $\hat{\psi}$, $\mathrm{e}^{2\hat{k}}$) into

$$\begin{aligned} \hat{\mathcal{E}}' &= \hat{\mathcal{E}} + 2i\Omega\,(z + \hat{\mathcal{F}} + \hat{\psi}\hat{\phi}) - \Omega^2(\rho^2/\hat{f} + \hat{\phi}^2)\,, \\ \hat{\psi}' &= \hat{\psi} + i\Omega\hat{\phi}\,, \qquad \mathrm{e}^{2\hat{k}'} = (1 - \Omega^2\rho^2/\hat{f}^2)\,\mathrm{e}^{2\hat{k}}\,. \end{aligned} \tag{13}$$

From these one may write down the asymptotically flat potentials $\mathcal{E}'$, $\psi'$, from which the rotating metric $g'_{\mu\nu}$ and the rotating electromagnetic potentials $A'_\mu$ may be obtained by solving duality equations. I give only here the expressions of the functions entering the Weyl form of this metric (again for $\gamma = 1$):

$$\begin{aligned} f' &= (|\hat{F}|^2/|\hat{F}'|^2)\lambda\, f\,, \qquad \mathrm{e}^{2k'} = \lambda\,\mathrm{e}^{2k}\,, \\ \partial_m\omega' &= \Omega^{-1}|\hat{F}'|^2\partial_m\lambda^{-1} - (2\rho/\hat{f})\lambda^{-1}\epsilon_{mn}\mathrm{Im}(\bar{\hat{F}}'\,\partial_n\hat{F}')\,, \end{aligned} \tag{14}$$

with $|\hat{F}'| \equiv (1/2)|1 - \hat{\mathcal{E}}' + 2\hat{\psi}'| = 1/|F'|$, $\lambda \equiv (1 - \Omega^2\rho^2/\hat{f}^2)$.

What are the general properties of the rotating Einstein–Maxwell fields thus generated? It can easily be checked that they are regular on the axis $\rho = 0$,

$$\mathrm{e}^{2k'} = 0\,, \qquad \partial_z\omega' = 0\,, \tag{15}$$

if the original static fields are regular. Another obvious property is the existence of stationary limit surfaces $f'(\rho, z) = 0$ for $\hat{f}(\rho, z) = \pm\Omega\rho$. Near such a zero of $f'$, the rotating metric

$$ds'^2 \simeq \mp 2\rho\, dt\, d\varphi \mp (\hat{F}'^2/\Omega\rho)(\mathrm{e}^{2k}(d\rho^2 + dz^2) + \rho^2\, d\varphi^2) \tag{16}$$

is non–degenerate.

It also seems that the horizons of the static solution (zeroes of $f$) generically carry over to the resulting rotating solutions, although a fully convincing proof is

not available at present (Weyl coordinates are not well adapted to this purpose). An illustration of this horizon conservation is the fact that the transformation $\Sigma$ (with arbitrary $\Omega$ and $\gamma$) transforms an extreme Reissner–Nordström black hole ($M^2 = Q^2$) into an extreme Kerr–Newman black hole ($M'^2 = Q'^2 + a'^2$).

The rotating metric (14) may present Kerr–like ring singularities corresponding to the zeroes of the function $|\hat{F}'|^2(\rho, z)$. In the plane–symmetric case, these rings are located in the plane $z = 0$ ($\mathrm{Im}\hat{F}' = 0$) with radii given by the solutions of the equation

$$2\,\mathrm{Re}\hat{F}' = (1 + \hat{\psi})^2 - \hat{f}' = 0 \qquad \mathrm{for} z = 0 \tag{17}$$

($\hat{f}' \equiv \hat{f} - \Omega^2\rho^2/\hat{f}$). In the case of the solutions studied in [10], these unwanted ring singularities may be avoided by suitably choosing the parameters of the static solution and/or the parameter $\Omega$.

The transformation $\Sigma$ does not generically commute with SU(2,1) transformations. This raises the question of classifying inequivalent solutions generated from a given static solution by transformations $\mathcal{U}\Sigma\mathcal{U}^{-1}$, with $\mathcal{U} \in \mathrm{SU}(2,1)$. A related question is that of the precise connection of the direct spin–generating method presented here with other spin–generating techniques.

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